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On the uniqueness of the coincidence index on orientable differentiable manifolds

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Abstract

The fixed point index of topological fixed point theory is a well studied integer-valued algebraic invariant of a mapping which can be characterized by a small set of axioms. The coincidence index is an extension of the concept to topological (Nielsen) coincidence theory. We demonstrate that three natural axioms are sufficient to characterize the coincidence index in the setting of continuous mappings on oriented differentiable manifolds, the most common setting for Nielsen coincidence theory.

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1. Introduction

For two mappings $f, g : X \rightarrow Y$, we say that $x \in X$ is a *coincidence point* of f and g (we write $x \in \text{Coin}(f, g)$) if $f(x) = g(x)$. This notion generalizes the common concept of a fixed point, in which $X = Y$ and $g = \text{id}$, the identity mapping. For any open set $U \subset X$, let $\text{Coin}(f, g, U) = \text{Coin}(f, g) \cap U$.

The well-known fixed point index of a mapping over a given subset (see [1] or [2]) was generalized to Nielsen coincidence theory by Schirmer [3] in the setting of mappings between oriented manifolds of the same (finite) dimension. Although Nielsen coincidence theory of nonmanifolds [4] and possibly nonorientable manifolds of nonequal dimension has been studied, very little is known, and the coincidence index is in general undefined in those settings.

Since definitions of the fixed point index of a mapping on a set can be cumbersome, it is often presented in the expository literature in terms of several properties: it is homotopy invariant, is additive on disjoint subsets, and the total index over the whole space equals the Lefschetz number of the mapping. This presentation is bolstered by uniqueness results which use such properties as axioms, and demonstrate that at most one function can satisfy the axioms (see [5], in which five axioms are used).

In [6], Furi et al. show that three axioms: homotopy invariance, additivity, and a normalization axiom, are sufficient to uniquely characterize the fixed point index in the setting of (possibly nonoriented) differentiable manifolds. We

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prove a similar result in coincidence theory, though we additionally will require our manifolds to be orientable, and we require a stronger normalization axiom.

The approach, following the structure of [6], is to first demonstrate the uniqueness of the coincidence index for mappings $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$. The uniqueness result on orientable manifolds is then obtained by approximating our mappings first by differentiable, and then (locally) by linear maps. The last section of the paper is devoted to the distinction between our normalization axiom and the weaker axiom of [6]. We show that a uniqueness result can be obtained with the weak normalization axiom in the special case of coincidence theory of selfmaps, but that this uniqueness does not extend to the general setting.

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2. Preliminaries

Throughout the paper, X and Y will be connected oriented differentiable manifolds of some particular dimension n . For continuous maps $f, g : X \rightarrow Y$ and some open subset $U \subset X$, we say that the triple (f, g, U) is *admissible* if the set of coincidences of f and g in U is compact. In particular, this will be true provided that U has compact closure and no coincidences on its topological boundary. Let $\mathcal{C}(X, Y)$ be the set of all admissible triples of maps from X to Y .

If $f_t, g_t : X \times [0, 1] \rightarrow Y$ are homotopies and $U \subset X$ is an open subset, we say that the pair (f_t, g_t) is a pair of *admissible homotopies in U* if

$$\{(x, t) \mid x \in \text{Coin}(f_t, g_t, U), t \in [0, 1]\}$$

is a compact subset of $X \times [0, 1]$. In this case we say that (f_0, g_0, U) is *admissibly homotopic* to (f_1, g_1, U) .

Let $D_X : H^{n-q}(X) \rightarrow H_q(X)$ and $D_Y : H^{n-q}(Y) \rightarrow H_q(Y)$ be the Poincaré duality isomorphisms. Given maps $f, g : X \rightarrow Y$, consider the composition

$$H_q(X) \xrightarrow{f_q} H_q(Y) \xrightarrow{D_Y^{-1}} H^{n-q}(Y) \xrightarrow{g^{n-q}} H^{n-q}(X) \xrightarrow{D_X} H_q(X),$$

where f_q is the map induced in dimension q homology (with rational coefficients) by f , and g^{n-q} is the map induced in dimension $n - q$ cohomology by g . Then the *Lefschetz number* of f and g is defined as

$$L(f, g) = \sum_{q=0}^n (-1)^q \text{tr}(D_X \circ g^{n-q} \circ D_Y^{-1} \circ f_q).$$

Our main result concerns the uniqueness of the coincidence index with respect to the following axioms. Throughout, ind denotes a function $\text{ind} : \mathcal{C}(X, Y) \rightarrow \mathbb{R}$.

Axiom 1 (*Additivity axiom*). Given $(f, g, U) \in \mathcal{C}(X, Y)$, if U_1 and U_2 are disjoint open subsets of U with $\text{Coin}(f, g, U) \subset U_1 \cup U_2$, then

$$\text{ind}(f, g, U) = \text{ind}(f, g, U_1) + \text{ind}(f, g, U_2).$$

Axiom 2 (*Homotopy axiom*). If $(f_0, g_0, U) \in \mathcal{C}(X, Y)$ and $(f_1, g_1, U) \in \mathcal{C}(X, Y)$ are admissibly homotopic, then

$$\text{ind}(f_0, g_0, U) = \text{ind}(f_1, g_1, U).$$

Axiom 3 (*Normalization axiom*). For $(f, g, X) \in \mathcal{C}(X, Y)$, we have

$$\text{ind}(f, g, X) = L(f, g).$$

Axiom 4 (*Weak normalization axiom*). If $(f, \text{id}, X) \in \mathcal{C}(X, X)$, with f a constant map and id the identity map, then $\text{ind}(f, \text{id}, X) = 1$.

The weak normalization axiom is a special case of the normalization axiom where f and g are selfmaps, since a straightforward calculation will show that $L(f, \text{id}) = 1$ when f is constant. The full normalization axiom is needed to handle coincidence theory of non-selfmaps, in which case the weak normalization axiom will not apply. Many of our results, however, require only the weak version, and we will be careful to indicate in our hypotheses which normalization axiom is being used.

Our main result is the following:

Theorem 5. *There is at most one function $\text{ind}: \mathcal{C}(X, Y) \rightarrow \mathbb{R}$ satisfying the additivity, homotopy, and normalization axioms.*

This result is a slightly weakened generalization of the main result of [6], which demonstrates the uniqueness of the fixed point axiom with respect to the additivity, homotopy, and weak normalization axioms.

We can derive some immediate corollaries from these three axioms, which will be useful in the exposition to follow. In the following propositions, $\text{ind}: \mathcal{C}(X, Y) \rightarrow \mathbb{R}$ denotes any function which satisfies the homotopy axiom, additivity axiom, and, whenever $X = Y$, the weak normalization axiom.

Proposition 6 (Fixed-point property). *If $(f, \text{id}, U) \in \mathcal{C}(X, X)$, then*

$$\text{ind}(f, \text{id}, U) = i(f, U),$$

where the right-hand side is the well-known fixed point index.

Proof. To demonstrate that $\text{ind}(f, \text{id}, U)$ is the fixed point index, we need only show that it obeys the three axioms described in [6]. Each of those three axioms are clearly special cases of our homotopy, additivity, and weak normalization axioms, so they are satisfied. \square

Proposition 7 (Empty set property). *The triple (f, g, \emptyset) is admissible, and has index 0.*

Proof. By the additivity axiom, we write $\emptyset = \emptyset \cup \emptyset$, and we have

$$\text{ind}(f, g, \emptyset) = \text{ind}(f, g, \emptyset) + \text{ind}(f, g, \emptyset),$$

and so $\text{ind}(f, g, \emptyset) = 0$. \square

Proposition 8 (Excision property). *If $(f, g, U) \in \mathcal{C}(X, Y)$ and $U' \subset U$ is an open subset containing $\text{Coin}(f, g, U)$, then $\text{ind}(f, g, U) = \text{ind}(f, g, U')$.*

Proof. By the additivity axiom, we have:

$$\text{ind}(f, g, U) = \text{ind}(f, g, U') + \text{ind}(f, g, \emptyset) = \text{ind}(f, g, U'). \quad \square$$

Proposition 9 (Solution property). *If $\text{ind}(f, g, U) \neq 0$, then f and g have a coincidence on U .*

Proof. We prove the contrapositive: if f and g have no coincidence on U , then by the excision property we have

$$\text{ind}(f, g, U) = \text{ind}(f, g, \emptyset) = 0. \quad \square$$

3. The coincidence index for mappings of \mathbb{R}^n

Let M_n be the space of linear maps $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$, and Gl_n the subspace of invertible maps. Define the set $N \subset M_n \times M_n$ as the set of all pairs of maps (A, B) such that $B - A \in Gl_n$. Note that N gives precisely the linear mappings for which $(A, B, \mathbb{R}^n) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$, since $\text{Coin}(A, B)$ will be a linear subspace of \mathbb{R}^n , and thus will be compact if and only if it is $\{0\}$, in which case $\det(B - A) \neq 0$.

We define

$$N^+ = \{(A, B) \mid \det(B - A) > 0\},$$

$$N^- = \{(A, B) \mid \det(B - A) < 0\},$$

and note that N is a disconnected set, with components N^+ and N^- . Each of these is open in $M_n \times M_n$, and therefore path-connected. Thus if some function $\text{ind}: \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies the homotopy axiom, then for any $(A, B) \in M_n \times M_n$, the value of $\text{ind}(A, B, \mathbb{R}^n)$ depends only on the path component containing (A, B) .

If ind additionally satisfies the weak normalization axiom, then $\text{ind}(0, \text{id}, \mathbb{R}^n) = 1$, and so we have that $\text{ind}(A, B, \mathbb{R}^n) = 1$ for all $(A, B) \in N^+$.

Lemma 10. *For any $(A, B) \in N^-$, if $\text{ind}: \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies the additivity, homotopy, and weak normalization axioms, we have*

$$\text{ind}(A, B, \mathbb{R}^n) = -1.$$

Proof. It suffices to show that $\text{ind}(A, B, \mathbb{R}^n) = -1$ for some particular pair $(\hat{A}, \hat{B}) \in N^-$. We will show this for $\hat{B} = \text{id}$ and \hat{A} the linear map of the same name described in Lemma 3.1 of [6]:

$$\hat{A}(x_1, \dots, x_{n-1}, x_n) = (0, \dots, 0, 2x_n).$$

Since we are taking \hat{B} to be the identity map, Proposition 6 implies that we need only demonstrate that the fixed point index of \hat{A} is -1 , and this is demonstrated in Lemma 3.1 of [6]. \square

The above discussion and lemma give our first uniqueness result.

Lemma 11. *For any $\text{ind}: \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ satisfying the additivity, homotopy, and weak normalization axioms, if $A, B: \mathbb{R}^n \rightarrow \mathbb{R}^n$ are linear maps and $(A, B, \mathbb{R}^n) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$, then*

$$\text{ind}(A, B, \mathbb{R}^n) = \text{sign } \det(B - A).$$

For differentiable maps $f, g: X \rightarrow Y$, let $df_x, dg_x: \mathbb{R}^n \rightarrow \mathbb{R}^n$ denote the derivative maps of f and g at the point $x \in X$. If the triple (f, g, U) is admissible, and $dg_p - df_p \in Gl_n$ for every $p \in \text{Coin}(f, g, U)$, then we say that the triple (f, g, U) is *nondegenerate*.

Since we have established that the index is very well behaved for linear mappings, the following linearization result for the index is very useful.

Lemma 12. *Let $(f, g, U) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ be a nondegenerate triple. Then each coincidence point p of f and g is isolated, and for some isolating neighborhood V of p , we have*

$$\text{ind}(f, g, V) = \text{ind}(df_p, dg_p, \mathbb{R}^n),$$

where $\text{ind}: \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ is any function satisfying the additivity, homotopy, and weak normalization axioms.

Proof. Let p be a coincidence point. Since f and g are differentiable at p , we have

$$f(x) = f(p) + df_p(x - p) + |x - p|\varepsilon(x - p),$$

$$g(x) = g(p) + dg_p(x - p) + |x - p|\delta(x - p),$$

where $\varepsilon, \delta: U \rightarrow \mathbb{R}^n$ are continuous with $\varepsilon(0) = \delta(0) = 0$. Then we have

$$\begin{aligned} |g(x) - f(x)| &= |(dg_p - df_p)(x - p) + |x - p|(\delta(x - p) - \varepsilon(x - p))| \\ &\geq |(dg_p - df_p)(x - p)| - |x - p||\delta(x - p) - \varepsilon(x - p)|. \end{aligned}$$

Since $dg_p - df_p$ is a linear map, we have

$$|(dg_p - df_p)(x - p)| = |x - p| \left| (dg_p - df_p) \left(\frac{x - p}{|x - p|} \right) \right| \geq |x - p| \left(\inf_{|v|=1} |(dg_p - df_p)(v)| \right),$$

and so

$$|g(x) - f(x)| \geq |x - p| \left(\inf_{|v|=1} |(dg_p - df_p)(v)| - |\delta(x - p) - \varepsilon(x - p)| \right).$$

The infimum is known to be strictly positive, since (f, g, U) is nondegenerate. Thus for all x sufficiently close to p , the term $|\delta(x - p) - \varepsilon(x - p)|$ will be sufficiently small so that $|g(x) - f(x)| > 0$, which is to say that p is an isolated coincidence point.

It remains to show that if V is an isolating neighborhood of p , then

$$\text{ind}(f, g, V) = \text{ind}(df_p, dg_p, \mathbb{R}^n).$$

Define the following homotopies:

$$f_t(x) = f(p) + df_p(x - p) + t|x - p|\varepsilon(x - p),$$

$$g_t(x) = g(p) + dg_p(x - p) + t|x - p|\delta(x - p).$$

By the same argument as above, for all $t \in [0, 1]$ we have

$$|g_t(x) - f_t(x)| > 0$$

for all x in some neighborhood $W \subset V$ of p . Thus (f_t, g_t) is an admissible homotopy on W of f and g to the affine linear maps $x \mapsto f(p) + df_p(x - p)$ and $x \mapsto g(p) + dg_p(x - p)$. These affine linear maps have p as their only coincidence point, since $dg_p - df_p \in Gl_n$. Thus we have

$$\begin{aligned} \text{ind}(f, g, W) &= \text{ind}(f(p) + df_p(x - p), g(p) + dg_p(x - p), W) \\ &= \text{ind}(f(p) + df_p(x - p), g(p) + dg_p(x - p), \mathbb{R}^n) \end{aligned}$$

by the homotopy axiom and the excision property.

But these affine maps are clearly admissibly homotopic to df_p and dg_p , and so we have

$$\text{ind}(f, g, V) = \text{ind}(f, g, W) = \text{ind}(df_p, dg_p, \mathbb{R}^n),$$

where the first equality is by the excision property, and the second is by the homotopy axiom. \square

4. The coincidence index for mappings of orientable manifolds

Lemmas 11 and 12 established the uniqueness of the index for nondegenerate pairs of mappings on \mathbb{R}^n with respect to the additivity, homotopy, and weak normalization axioms. The next two lemmas show that any index for nondegenerate pairs on arbitrary orientable manifolds is computed similarly to the index of Lemma 11.

Lemma 13. *If $(f, g, U) \in \mathcal{C}(X, Y)$ is a nondegenerate triple, then each coincidence point is isolated.*

Proof. For some $p \in \text{Coin}(f, g, U)$, choose a chart $Z_q \subset Y$ containing $q = f(p) = g(p)$ with a diffeomorphism $\psi_q : Z_q \rightarrow \mathbb{R}^n$. Choose a chart $W_p \subset f^{-1}(Z_q) \cap g^{-1}(Z_q)$ containing p with a diffeomorphism $\phi_p : W_p \rightarrow \mathbb{R}^n$. Define $\omega : \mathcal{C}(W_p, X) \rightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ by

$$\omega(f, g, W_p) = (\psi_q \circ f \circ \phi_p^{-1}, \psi_q \circ g \circ \phi_p^{-1}, \phi_p(W_p)).$$

Since (f, g, U) is nondegenerate, then clearly (f, g, W_p) will be nondegenerate. For any coincidence point $x \in \text{Coin}(\omega(f, g, W_p))$, we have

$$d(\psi_q \circ f \circ \phi_p^{-1})_x - d(\psi_q \circ g \circ \phi_p^{-1})_x = d(\psi_q)_{f(\phi_p^{-1}(x))} (df_{\phi_p^{-1}(x)} - dg_{\phi_p^{-1}(x)}) d(\phi_p^{-1})_x,$$

since $f(\phi_p^{-1}(x)) = g(\phi_p^{-1}(x))$. Since ϕ_p and ψ_q are diffeomorphisms, the above will be in Gl_n by nondegeneracy of (f, g, W_p) . Thus $\omega(f, g, W_p)$ is nondegenerate, which implies by Lemma 12 that $\text{Coin}(\omega(f, g, W_p))$ is a set of isolated points. But this coincidence set is in diffeomorphic correspondence via ϕ_p and ψ_q to $\text{Coin}(f, g, W_p)$, so this too is a set of isolated points, and in particular p is an isolated coincidence point. \square

We note that the next lemma contains our only use of the full (rather than the weak) normalization axiom, and also our only use of the orientability hypothesis on the manifolds X and Y .

Lemma 14. *If $(f, g, U) \in \mathcal{C}(X, Y)$ is a nondegenerate triple, and $\text{ind} : \mathcal{C}(X, Y) \rightarrow \mathbb{R}$ satisfies the additivity, homotopy, and normalization axioms, then*

$$\text{ind}(f, g, U) = \sum_{p \in \text{Coin}(f, g, U)} \text{sign}(\det(dg_p - df_p)). \quad (1)$$

Proof. Note that since (f, g, U) is nondegenerate, by Lemma 13 the coincidence set $C = \text{Coin}(f, g, U)$ consists of finitely many isolated points.

First we prove the theorem in the special case where $X = Y = \mathbb{R}^n$. For each coincidence point $p \in U$, let V_p be an isolating neighborhood of p . Since C is compact, we may choose the sets V_p to be pairwise disjoint. Then by the additivity property, Lemmas 12 and 11, we have

$$\text{ind}(f, g, U) = \sum_{p \in C} \text{ind}(f, g, V_p) = \sum_{p \in C} \text{ind}(df_p, dg_p, \mathbb{R}^n) = \sum_{p \in C} \text{sign}(\det(dg_p - df_p)).$$

It remains to prove the result in the general case that X and Y are not both \mathbb{R}^n . About each isolated coincidence point p , choose pairwise disjoint isolating neighborhoods W_p diffeomorphic to \mathbb{R}^n by orientation preserving diffeomorphisms $\phi_p : W_p \rightarrow \mathbb{R}^n$. We also choose neighborhoods Z_q of each coincidence value $q = f(p) = g(p)$ with Z_q diffeomorphic to \mathbb{R}^n by orientation preserving diffeomorphisms $\psi_q : Z_q \rightarrow \mathbb{R}^n$. Let $\omega : \mathcal{C}(W_p, Y) \rightarrow \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ be as in Lemma 13.

Note that ω has an inverse given by

$$\omega^{-1}(F, G, S) = (\psi_q^{-1} \circ F \circ \phi_p, \psi_q^{-1} \circ G \circ \phi_p, \phi_p^{-1}(S)),$$

where $(F, G, S) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$. Then we have trivially that

$$\text{ind}(f, g, W_p) = \text{ind} \circ \omega^{-1}(\omega(f, g, W_p)).$$

We now note that $\text{ind} \circ \omega^{-1} : \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies the additivity, homotopy, and normalization axioms. The additivity and homotopy axioms are clear, but a brief calculation is needed for the normalization axiom. We must show that $\text{ind} \circ \omega^{-1}(F, G, \mathbb{R}^n) = L(F, G)$.

We have

$$\text{ind}(\omega^{-1}(F, G, \mathbb{R}^n)) = \text{ind}(\psi_q^{-1} \circ F \circ \phi_p, \psi_q^{-1} \circ G \circ \phi_p, W_p) = L(\psi_q^{-1} \circ F \circ \phi_p, \psi_q^{-1} \circ G \circ \phi_p)$$

by the normalization axiom. Consider the diagram:

$$\begin{array}{ccccccc} H_*(W_p) & \xrightarrow{\phi_{p*}} & H_*(\mathbb{R}^n) & \xrightarrow{F_*} & H_*(\mathbb{R}^n) & \xrightarrow{\psi_{q*}^{-1}} & H_*(Z_q) \\ D_X^{-1} \uparrow & & D_{\mathbb{R}^n}^{-1} \uparrow & & D_{\mathbb{R}^n} \downarrow & & D_Y \downarrow \\ H^{n-*}(W_p) & \xleftarrow{\phi_p^*} & H^{n-*}(\mathbb{R}^n) & \xleftarrow{G^{n-*}} & H^{n-*}(\mathbb{R}^n) & \xleftarrow{\psi_q^{-1*}} & H^{n-*}(Z_q) \end{array}$$

where starred maps are the induced maps in (co)homology. The Lefschetz number above is the alternating sum of the traces of the maps formed by a clockwise walk around the perimeter of the diagram, starting at the upper left corner.

Note by the functoriality of the duality maps that the squares at right and left in the diagram will commute. Using this fact, along with cyclic permutation of maps inside the trace, we have

$$\begin{aligned} \text{tr}(D_X^{-1} \circ \phi_p^* \circ \psi_q^{-1*} \circ D_Y \circ \psi_{q*}^{-1} \circ F_* \circ \phi_{p*}) &= \text{tr}(D_X^{-1} \circ \phi_p^* \circ G^* \circ D_{\mathbb{R}^n} \circ F_* \circ \phi_{p*}) \\ &= \text{tr}(\phi_{p*} \circ D_X^{-1} \circ \phi_p^* \circ G^* \circ D_{\mathbb{R}^n} \circ F_*) \\ &= \text{tr}(D_{\mathbb{R}^n}^{-1} \circ G^* \circ D_{\mathbb{R}^n} \circ F_*), \end{aligned}$$

and so

$$L(\psi_q^{-1} \circ F \circ \phi_p, \psi_q^{-1} \circ G \circ \phi_p) = L(F, G),$$

which shows that $\text{ind} \circ \omega^{-1}$ satisfies the normalization axiom.

We have shown that $\text{ind} \circ \omega^{-1}$ is a real valued function on $\mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ which satisfies the three axioms. Thus, by the special case above, it is calculated according to (1).

Thus we have:

$$\begin{aligned}\operatorname{ind}(f, g, U) &= \sum_{p \in C} \operatorname{ind}(f, g, W_p) = \sum_{p \in C} \operatorname{ind} \circ \omega^{-1}(\omega(f, g, W_p)) \\ &= \sum_{p \in C} \operatorname{sign}(\det(d\psi_q \cdot dg_p \cdot d\phi_p^{-1} - d\psi_q \cdot df_p \cdot d\phi_p^{-1}))\end{aligned}$$

by the derivative chain rule. (For brevity, we have written $d\phi_p^{-1}$ to indicate $d(\phi_p^{-1})_p$ and $d\psi_q$ to indicate $d(\psi_q)_q$.) But the above is simply

$$\sum_{p \in C} \operatorname{sign}(\det(d\psi_q(dg_p - df_p)d\phi_p^{-1})) = \sum_{p \in C} \operatorname{sign}(\det(dg_p - df_p))$$

since all ϕ_p and ψ_q are taken to be orientation preserving. \square

One final lemma is required, showing that any pair of maps can be approximated by a nondegenerate pair of maps.

Lemma 15. *Let $(f, g, U) \in \mathcal{C}(X, Y)$, and $V \subset U$ be an open subset containing $\operatorname{Coin}(f, g, U)$ with compact closure $\bar{V} \subset U$. Then (f, g, V) is admissibly homotopic to a nondegenerate triple.*

Proof. In order to facilitate the construction of explicit straightline homotopies, we begin by embedding our manifolds in some Euclidean space, and approximating our maps accordingly by close polynomial approximations.

Without loss of generality, assume that X and Y are embedded in \mathbb{R}^k for some k . By the ε -neighborhood theorem (see [7]), there exists a neighborhood Ω of Y in \mathbb{R}^k with a submersive retraction $r : \Omega \rightarrow Y$ such that $|x - r(x)| = \operatorname{dist}(x, Y)$.

For any $\delta > 0$, by the Weierstrass approximation theorem, there exist polynomial maps $f_\delta, g_\delta : \mathbb{R}^k \rightarrow \mathbb{R}^k$ with $|f(x) - f_\delta(x)| < \delta$ and $|g(x) - g_\delta(x)| < \delta$ for all $x \in \bar{V}$. Since \bar{V} is compact, we may choose δ sufficiently small so that the homotopies

$$\begin{aligned}f_t(x) &= r((1-t)f(x) + tf_\delta(x)), \\ g_t(x) &= r((1-t)g(x) + tg_\delta(x))\end{aligned}$$

are well-defined and (f_t, g_t) is an admissible homotopy on V . Thus the pair (f, g) is admissibly homotopic to $(r \circ f_\delta, r \circ g_\delta)$ in V . To simplify our notation, let $f' = r \circ f_\delta$, and $g' = r \circ g_\delta$. We will show that the pair (f', g') is admissibly homotopic to some nondegenerate pair.

Let B be a ball about the origin sufficiently small that for all $x \in \bar{V}$ and $y \in B$ the function $x \mapsto r(f'(x) + y)$ is defined and has no coincidences with g' on ∂V . Then define $H : V \times B \rightarrow V \times Y$ by

$$H(x, y) = (x, r(f'(x) + y)),$$

and note that the derivative map

$$dH_{(x,y)} : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^k \times T_{r(f'(x)+y)}Y$$

is surjective (since r is a submersion). Thus at points where the image of H intersects with the graph of g' in $V \times Y$, the intersection will be transversal.

By the transversality theorem (see [7]), $H(x, y)$ is transversal to graph g' for almost all $y \in B$. Choose one such \bar{y} so that

$$\bar{H}(x) = (x, r(f'(x) + \bar{y}))$$

is transversal to graph g' . Note that a pair of mappings is nondegenerate if and only if their graphs intersect transversally. This means that $(r(f' + \bar{y}), g', \bar{V})$ is a nondegenerate triple.

But our assumption on B means that the homotopy given by

$$(r(f' + t\bar{y}), g')$$

is admissible on \bar{V} , and thus the nondegenerate triple above is admissibly homotopic to (f', g', \bar{V}) , which has already been shown to be admissibly homotopic to (f, g, V) . \square

Now we are ready to prove Theorem 5:

Proof. If $(f, g, U) \in \mathcal{C}(X, Y)$, then by Lemma 15 there is an open set $V \subset U$ containing $\text{Coin}(f, g, U)$ with (f, g, V) admissibly homotopic to a nondegenerate triple (f', g', V) . Thus by excision and the homotopy axiom, we have $\text{ind}(f, g, U) = \text{ind}(f', g', U)$, and so by Lemma 14 we have

$$\text{ind}(f, g, U) = \sum_{p \in \text{Coin}(f', g', V)} \text{sign}(\det(dg'_p - df'_p)).$$

The above calculation does not depend on the choice of nondegenerate triple (f', g', U) , since any alternative choice would automatically have the same index by the homotopy axiom. Since any coincidence index must obey the calculation given above, we have shown that all indices must agree, and that there is at most one. \square

At this point we will briefly address the issue of the existence of the coincidence index. The above lemmas could be used to define a coincidence index in the following way: given any admissible triple (f, g, U) , let (f', g', V) be a nondegenerate triple given by Lemma 15, admissibly homotopic to (f, g, V) with $\text{Coin}(f, g, U) \subset V$. We then define $\text{ind}(f, g, U) = \text{ind}(f', g', V)$, where $\text{ind}(f', g', V)$ is computed according to (1).

An index defined in this way would clearly satisfy our axioms, provided that it is well defined. We believe this to be the case, but a verification is apparently nontrivial, and would require a demonstration of:

Conjecture 16. *If (f, g, U) and (h, k, U) are admissibly homotopic nondegenerate triples, then*

$$\sum_{p \in \text{Coin}(f, g, U)} \text{sign}(\det(dg_p - df_p)) = \sum_{q \in \text{Coin}(h, k, U)} \text{sign}(\det(dk_q - dh_q)).$$

5. Uniqueness and the weak normalization axiom

It is natural to ask whether any meaningful version of the above theorem can be proved using the weak normalization axiom in place of the full normalization axiom. Since the weak normalization can only apply to selfmaps, an obvious question is: is there is a unique coincidence index on $\mathcal{C}(X, X) \rightarrow \mathbb{R}$ which satisfies the homotopy, additivity, and weak normalization axioms? We answer this question in the affirmative, and additionally show that orientability of X is not required in this setting.

Theorem 17. *For any particular (perhaps nonorientable) differentiable manifold X , there is at most one coincidence index $\text{ind}: \mathcal{C}(X, X) \rightarrow \mathbb{R}$ satisfying the additivity, homotopy, and weak normalization axioms.*

Proof. Note that the proof of Lemma 14 is the only place where either of the orientability hypothesis or the full (as opposed to the weak) normalization axiom is used. Thus our proof here may make use of any of our lemmas except for Lemma 14.

As in the proof of Theorem 5, our proof consists of deriving a formula for the computation of the index of some triple (f, g, U) . By Lemma 15 we may assume without loss of generality that (f, g, U) is nondegenerate. As in the proof of Lemma 14, an explicit formula for the index is clear in the special case where $X = \mathbb{R}^n$. In the case where X is not \mathbb{R}^n , we first change the triple (f, g, U) by a homotopy, and then follow similar steps to those used in Lemma 14.

For some isolated coincidence point p , let $q = f(p) = g(p)$. Let $\gamma: [0, 1] \rightarrow X$ be a path in X from q to p which avoids all other points of $\text{Coin}(f, g, U)$, and let V be a contractible neighborhood of γ with closure disjoint from $\text{Coin}(f, g, U)$. Since V is homeomorphic to an open ball in \mathbb{R}^n , there are homotopies f_t, g_t such that f_t and g_t agree for all t with f and g , respectively, on $U - V$, and $\text{Coin}(f_t, g_t, V) = \{p\}$ with $f_t(p) = g_t(p) = \gamma(t)$. Thus the triple (f_t, g_t, U) will be an admissible homotopy of the triple (f, g, U) to some triple (f_1, g_1, U) , with $f_1(p) = g_1(p) = p$.

In this way, we have converted the coincidence point at p into a fixed point at p without disturbing the behavior of f and g at the other coincidence points. By iterating the above construction for each coincidence point p , we obtain

an admissible triple (f', g', U) admissibly homotopic to (f, g, U) such that all points of $\text{Coin}(f', g', U)$ are actually fixed points of both f' and g' .

We now continue as in the proof of Lemma 14. About each isolated coincidence point $p \in \text{Coin}(f', g', U)$, choose pairwise disjoint isolating neighborhoods W_p diffeomorphic to \mathbb{R}^n by diffeomorphisms $\phi_p: W_p \rightarrow \mathbb{R}^n$. Letting $V_p = f'^{-1}(W_p) \cap g'^{-1}(W_p) \cap W_p$, we will define $\omega(f', g', V_p) \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ as in Lemma 13, only this time our two charts can be taken to be the same, since p is a fixed point. Let

$$\omega(f', g', V_p) = (\phi_p \circ f' \circ \phi_p^{-1}, \phi_p \circ g' \circ \phi_p^{-1}, \phi(V_p)),$$

and as in Lemma 14, we note that ω has an inverse given by

$$\omega^{-1}(F, G, S) = (\phi_p^{-1} \circ F \circ \phi_p, \phi_p^{-1} \circ G \circ \phi_p, V_p),$$

and trivially we have $\text{ind}(f', g', V_p) = \text{ind} \circ \omega^{-1}(\omega(f', g', V_p))$.

Now we note that $\text{ind} \circ \omega^{-1}: \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies the additivity, homotopy, and weak normalization axioms. The additivity and homotopy axioms are clear, and the weak normalization is not difficult: if F is the constant function $F(x) = c$ and G is the identity, then

$$\text{ind} \circ \omega^{-1}(F, G, S) = \text{ind}(\phi^{-1}(c), \text{id}, V_p),$$

and this equals 1 by the weak normalization axiom.

Thus $\text{ind} \circ \omega^{-1}$ satisfies the additivity, homotopy, and weak normalization axioms, and so is the coincidence index by our special case above for $X = \mathbb{R}^n$. Thus, letting $C = \text{Coin}(f', g', U)$, we have

$$\begin{aligned} \text{ind}(f, g, U) &= \text{ind}(f', g', U) = \sum_{p \in C} \text{ind}(f', g', V_p) = \sum_{p \in C} \text{ind} \circ \omega^{-1}(\omega(f', g', V_p)) = \sum_{p \in C} \text{ind}(\omega(f', g', V_p)) \\ &= \sum_{p \in C} \text{sign}(\det(d\phi_p \cdot dg'_p \cdot d\phi_p^{-1} - d\phi_p \cdot df'_p \cdot d\phi_p^{-1})) = \sum_{p \in C} \text{sign}(\det(dg'_p - df'_p)). \end{aligned}$$

By the homotopy axiom, this formula is independent of the choice of the admissible homotopy to (f', g', U) , and the uniqueness is shown. \square

The above theorem could be restated as follows: If \mathcal{C}_s is the class of *all* admissible triples of selfmaps on differentiable manifolds, then there is a unique coincidence index $\text{ind}: \mathcal{C}_s \rightarrow \mathbb{R}$ which satisfies the additivity, homotopy, and weak normalization axioms. This index, for any triple (f, g, U) , would be defined by using the unique index defined on $\mathcal{C}(X, X)$, where X is the domain of f and g . Our above theorem, in this sense, can be seen as a direct generalization of the result of [6] to coincidence theory of selfmaps.

A natural question to ask is whether a further extension of [6] can be made to non-selfmaps as follows: If \mathcal{C} is the set of *all* admissible triples (f, g, U) where f and g are maps between orientable differentiable manifolds of the same dimension, is a coincidence index satisfying the additivity and homotopy axioms, which additionally satisfies the weak normalization axiom whenever f and g are selfmaps, unique? Such a uniqueness result would be stronger than our Theorem 5 but is false as the following example illustrates.

Example 18. Let \mathcal{C} be as above, the set of all admissible triples. Letting $\text{Ind}_{(X,Y)}$ denote the unique coincidence index on $\mathcal{C}(X, Y)$ given by Theorem 5, we can define a single index $\text{Ind}: \mathcal{C} \rightarrow \mathbb{R}$ by $\text{Ind}(f, g, U) = \text{Ind}_{(X,Y)}(f, g, U)$ when $(f, g, U) \in \mathcal{C}(X, Y)$. This gives a single coincidence index Ind on \mathcal{C} which satisfies the additivity, homotopy, and normalization axioms.

For any $c \in \mathbb{R}$, define $i_c: \mathcal{C} \rightarrow \mathbb{R}$ as follows:

$$i_c(f, g, U) = \begin{cases} \text{Ind}(f, g, U) & \text{if } f, g \text{ are selfmaps,} \\ c \cdot \text{Ind}(f, g, U) & \text{otherwise.} \end{cases}$$

For any value of c , this function clearly satisfies the additivity and homotopy axioms, and also satisfies the weak normalization axiom in the case when f and g are selfmaps.

Variation of the parameter c in the above example will produce many distinct “indices” on \mathcal{C} which satisfy the additivity, homotopy, and (when applicable) weak normalization axioms. Thus Theorems 5 and 17 seem to be the best uniqueness results available in our setting.

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